1. *Let K be the set of functions from $\{0,1\}$ to $\mathbb{N}$. Show that K is countable. Let M be the set of functions from $\mathbb{N}$ to $\mathbb{N}$. Show that M is uncountable.*

   Solution: Let $T : K \longrightarrow \mathbb{N}$ such that $T(f) = 2^{f(0)}3^{f(1)}$. Then by prime factorization theorem T is injective. Therefore there is a bijection from Range of $T \subset \mathbb{N}$ to K. Hence K is countable.

   Now, let us prove that M is uncountable. Assume that there exists an F which is a bijection from $\mathbb{N}$ to M. Let $g \in M$ defined by $g(n) = (F(n))(n) + 1$. Then, $g \neq F(n)$ for every $n \in \mathbb{N}$. Hence, g does not belong to the range of F, which is a contradiction. $\square$

2. *Let $\{v_n\}$ be the sequence defined by $v_1 = 1$ and $v_{n+1} = (v_n^2 + \frac{1}{2^n})^{\frac{1}{2}}$ for $n \geq 1$. Show that $\lim_{n\to\infty} v_n$ exists. Find the limit.*

   Solution: Since $v_{n+1} = (v_n^2 + \frac{1}{2^n})^{\frac{1}{2}}$ we have $v_{n+1}^2 = v_n^2 + \frac{1}{2^n}$.
   Therefore it is clear that $v_{n+1}^2 = \Sigma_{k=0}^{n} \frac{1}{2^k}$
   So $v_{n+1} = (\Sigma_{k=0}^{n} \frac{1}{2^k})^{\frac{1}{2}}$
   This clearly converges since it is a geometric series with common ratio $1/2$ which is less than 1 and $\lim_{n\to\infty} v_n = (\Sigma_{k=0}^{\infty} \frac{1}{2^n})^{\frac{1}{2}} = 2^{\frac{1}{2}}$ $\square$

3. *Let $f, g : \mathbb{R} \to \mathbb{R}$ be continuous functions. Show that $h : \mathbb{R} \to \mathbb{R}$ defined by $h(x) = \min\{f(x), g(x)\}$ is a continuous function. Show that the converse is not true.*

   Show that if f, g are differentiable at c and $f(c) \neq g(c)$ then h is differentiable at c.

   Solution: $h(x) = \min\{f(x), g(x)\} = \frac{f(x)+g(x)}{2} - \frac{|f(x)-g(x)|}{2}$. Since linear combination and modulus of continuous functions are continuous, h is continuous.
   To show that converse is not true,

   let $f(x) = \begin{cases} 0 & if\ x \leq 0 \\ 1 & if\ x > 0 \end{cases}$

   and let $g(x) = \begin{cases} 1 & if\ x \leq 0 \\ 0 & if\ x > 0 \end{cases}$

   Then h is constant zero function, which is continuous on $\mathbb{R}$ whereas f and g are not continuous at zero.

   Since $f(c) \neq g(c)$,with out loss of generality we can assume that $f(c) < g(c)$. Since they are continuous there exists an open interval around c , say U such that $f(x) < g(x)$ for every $x \in U$. Therefore $h(x) = \min\{f(x), g(x)\} = f(x)$ on U. Since f is differentiable at $c \in U$, so is h. $\square$

4. *Let $m : (0,1) \to \mathbb{R}$ be a continuous function. Suppose $\{m(x) : x \in \mathbb{R}\} \subseteq \mathbb{N}$. Show that m is a constant function.*

   Solution: Suppose m is not constant, then there exists x and y in $\mathbb{R}$ such that $m(x) + 1 \leq m(y)$
   Then by intermediate value theorem there exists a z between x and y such that $m(z) = m(x) + 1/2$.
   This will contradict the fact that range of m is contained in $\mathbb{N}$. $\square$

5. *Suppose $k \in \mathbb{N}$ and $B_1, \cdots, B_k$ are strictly positive. Show that*

   (a) $\lim_{n\to\infty} k^{\frac{1}{n}} = 1$.

   (b) $\lim_{n\to\infty}({B_1}^n + {B_2}^n + ... + {B_k}^n)^{\frac{1}{n}} = B$ *where* $B = Max\{B_j : 1 \le j \le k\}$

   Solution : (a) Let $\varepsilon > 0$ be arbitrary. From archimedes principle, one can find $N \in \mathbb{N}$ such that $k \le 1 + n\varepsilon$ for $n \ge N$. Now, $1 \le k \le (1+\varepsilon)^n$ for every $n \ge N$, since $1 + n\varepsilon \le (1+\varepsilon)^n$. Then, $1 \le k^{\frac{1}{n}} \le 1 + \varepsilon$ for every $n \ge N$.

   Therefore $|k^{\frac{1}{n}} - 1| \le \varepsilon$ for every $n \ge N$. Hence $\lim_{n\to\infty} k^{\frac{1}{n}} = 1$.

   (b) Since $B_i \ge 0$, we have $B^n \le ({B_1}^n + {B_2}^n + ... + {B_k}^n) \le (kB^n)$
   Therefore $B \le ({B_1}^n + {B_2}^n + ... + {B_k}^n)^{\frac{1}{n}} \le (kB^n)^{\frac{1}{n}}$
   Taking limit on both inequalities and using (a) we get $\lim_{n\to\infty}({B_1}^n + {B_2}^n + ... + {B_k}^n)^{\frac{1}{n}} = B$. $\square$

6. *Let a, b be real numbers with $a < b$, and let $f : [a, b] \to \mathbb{R}$ be a continuous function. Suppose f is differentiable (a, b) and $f'(x) \ne 0$ for every $x \in (a, b)$. 1)Show that f is one to one.*
   2) Show that either $f'(x) > 0$ for all $x \in (a, b)$ or $f'(x) < 0$ for all $x \in (a, b)$.

   Solution : It is enough to prove that f is monotonically increasing or decreasing. If not, let us assume that there exists x,y,$z \in \mathbb{R}$ with $x < y < z$ such that

$$f(x) < f(y) \;\; and \;\; f(z) < f(y) \;\; or \tag{1}$$

$$f(x) > f(y) \;\; and \;\; f(y) < f(z). \tag{2}$$

   Since f is continuous on $[x, y]$ it attains maximum and minimum. Let

$$f(\alpha) = \sup_{z\in[x,y]} f(z) \tag{3}$$

$$f(\beta) = \inf_{z\in[x,y]} f(z) \tag{4}$$

   If (1) happens α will not be x or y. Similarly if (2) happens then β will not be x or y. In either cases f attains a local maximum or minimum at α or β, which will contradict the fact that $f'(x) \ne 0$ for every $x \in [a, b]$. So f is monotonically increasing or decreasing. Hence f' is either strictly positive or negative.

7. *State and prove Taylor's theorem for real valued function on open subintervals of $\mathbb{R}$*

   Solution : **Statement of Theorem** : Suppose f is a real function on $[a, b]$, n is a positive integer, f^{n-1} is continuous on $[a, b]$, $f^n(t)$ exists for every $t \in (a, b)$. Let α, β be distinct points of $[a, b]$, and define $P(t) = \Sigma_{k=0}^{n-1} \frac{f^k(\alpha)}{k!}(t-\alpha)^k$. Then there exists a point x between α and β such that $f(\beta) = P(\beta) + \frac{f^n(x)}{n!}(\beta - \alpha)^n$.

   Proof : Let M be the number defined by

$$f(\beta) = P(\beta) + M(\beta - \alpha)^n \tag{5}$$

and put

$$g(t) = f(t) - P(t) - M(t-\alpha)^n \quad (a \leq t \leq b). \tag{6}$$

We have to show that $n!M = f^n(x)$ for some x between α and β. By (5) and (6)

$$g^n(t) = f^n(t) - n!M \tag{7}$$

Hence the proof will be complete if we can show that $g^n(x) = 0$ for some x between α and β. Since $P^k(\alpha) = f^k(\alpha)$ for $k = 0, ..., n-1$, we have

$$g(\alpha) = g'(\alpha) = ... = g^{n-1}(\alpha) = 0 \tag{8}$$

Our choice of M shows that $g(\beta) = 0$, so that $g'(x_1) = 0$ for some x_1 between α and β, by the Mean value theorem. Since $g'(\alpha) = 0$, we conclude similarly that $g''(x_2) = 0$ for some x_2 between α and x_1. After n steps we arrive at the conclusion that $g^n(x_n) = 0$ for some x_n between α and x_{n-1}, that is, between α and β. $\square$